IV. "On the Locus of Singular Points and Lines which occur in connexion with the Theory of the Locus of Ultimate Intersections of a System of Surfaces." By M. J. M. HILL, M.A., Sc.D., Professor of Mathematics at University College, London. Communicated by Professor HENRICI, F.R.S. Received October 5, 1891.

(Abstract.)

Introduction.

In a paper "On the c- and p-Discriminants of Ordinary Integrable Differential Equations of the First Order," published in vol. 19 of the 'Proceedings of the London Mathematical Society,' the factors which occur in the c-discriminant of an equation of the form $f(x, y, c) = 0$, where $f(x, y, c)$ is a rational integral function of x, y, c, are determined analytically.

It is shown* that if $E = 0$ be the equation of the envelope locus of the curves $f(x, y, c) = 0$; if $N = 0$ be the equation of their node locus; if $C = 0$ be the equation of their cusp locus; then the factors of the discriminant are E, N^2, C^3.

The singularities considered are those whose forms depend on the terms of the second degree only, when the origin of coordinates is at the singular point.

The object of this paper is to extend these results to surfaces.

It is well known that if the equation of a system of surfaces contain arbitrary parameters, and if a locus of ultimate intersections exist, then there cannot be more than two independent parameters.

Hence the investigation falls naturally into two parts: the first is the case where there is only one independent parameter, and the second is the case where there are two.

The investigation given in this paper is limited to the case in which the equation is rational and integral both as regards the coordinates and the parameters.

PART I.

The Equation of the Surfaces is a Rational Integral Function of the Coordinates and one Arbitrary Parameter.

In the case in which there is only one arbitrary parameter each surface of the system intersects the consecutive surface in a curve

* The theorem was originally given by Professor Cayley, in the 'Messenger of Mathematics,' vol. 2, 1872, pp. 6–12.

whose equations are the equation of the surface and the equation obtained by differentiating it with regard to the parameter. These equations will be called the fundamental equations in this part. Hence each surface touches the envelope along a curve, which is called a characteristic. It is known that the equation of the envelope may be obtained by eliminating the parameter from the fundamental equations and equating a factor of the result to zero. But it frequently happens that there are other factors of the result (or discriminant as it will in future be called) which when equated to zero do not give the equation of the envelope.

Following out the same line of argument as that used in reference to a system of plane curves, it will be shown that these factors are connected with loci of singular points. Now if each surface have one singular point, then its coordinates may in general be expressed as functions of the parameter of the surface to which it belongs. Hence the locus of all the singular points of the surfaces of the system is a curve. Its equations, therefore, cannot be found by equating a factor of the discriminant to zero. But if each surface of the system have upon it a nodal line, then the locus of the nodal lines of all the surfaces is a surface, and it will be shown that its equation may be found by equating to zero a factor of the discriminant.

The singular points in space, the form of which depends only on the terms of the second order, when the origin of coordinates is taken at the singular point, are :

(i.) The conic node, where all the tangent lines to the surface lie on a cone of the second order.

(ii.) The biplanar node or binode. This is the particular case of the preceding, in which the tangent cone to the surface breaks up into two non-coincident planes. These planes are called the biplanes, and their intersection is called the edge of the binode.

(iii.) The uniplanar node or unode. This is the particular case of the conic node, in which the tangent cone breaks up into two coincident planes. The plane with which these planes coincide is called the uniplane.

It is shown that a surface cannot have upon it a curve at every point of which there is a conic node. Hence there are two varieties of nodal lines to be considered; the first, being such that every point is a binode, may be called a binodal line; and the second, being such that every point on it is a unode, may be called a unodal line.

It will be proved that if $E = 0$ be the equation of the envelope locus, $B = 0$ the equation of the locus of binodal lines, $U = 0$ the equation of the locus of unodal lines, then the factors of the discriminant are in general E, B^2, U^3.

This is the general theorem, but it is assumed in the course of the investigation, when the discriminant is being formed, that the fundamental equations are satisfied by only one value of the parameter at each point on the envelope locus or on a locus of binodal or unodal lines.

The investigation is accordingly carried a step further, and it is shown that if the fundamental equations are satisfied by two equal values of the parameter at points on an envelope locus, or on a locus of binodal or unodal lines, the factors of the discriminant are E^2, B^3, U^4.

The geometrical meaning of the condition that the fundamental equations are satisfied by two equal values of the parameter in the case of the envelope is that the line of contact of the envelope with each surface of the system counts three times over as a curve of intersection, instead of twice as in the ordinary case, or that two consecutive characteristics coincide. The meaning of the condition in the case of the loci of singular lines is that each of these loci is also an envelope.

The results are given in the following table :—

Description of locus.	Factor of discriminant corresponding to locus.	Number of values of parameter satisfying fundamental equations.
Envelope	E (Art. 1)	1
Locus of binodal lines ..	B^2 (Art. 5)	1
Locus of unodal lines ...	U^3 (Art. 6)	1
Envelope such that two consecutive characteristics coincide.	E^2 (Art. 7)	2 coinciding
Locus of binodal lines, which is also an envelope.	B^3 (Art. 8)	2 coinciding
Locus of unodal lines, which is also an envelope.	U^4 (Art. 9)	2 coinciding

Part II.

The Equation of the System of Surfaces is a Rational Integral Function of the Coordinates and two Arbitrary Parameters.

In the case in which there are two arbitrary parameters in the equation of the system of surfaces, the equation of the locus of ulti-

mate intersections is found by eliminating the parameters between this equation and the two equations obtained by differentiating it with regard to the parameters. These equations will in this part of the investigation be called the fundamental equations.

In general the locus of ultimate intersections is a surface, for the coordinates of each point on it can be expressed as functions of the two arbitrary parameters. The exceptional cases in which it is not a surface are enumerated at the end of the paper. These include the case where the equation of the system of surfaces is of the first degree in the parameters. Hence it will be supposed that the degree of the equation of the system of surfaces in the parameters is above the first.

In general also the locus of ultimate intersections possesses the envelope property, and the equation of the envelope is determined by equating the discriminant, or a factor of it, to zero.

If factors of the discriminant exist which, when equated to zero, give surfaces not possessing the envelope property, then, as in Part I, it is shown that these surfaces are connected with loci of singular points.

Now the locus of singular points of a system of surfaces whose equation contains two arbitrary parameters is in general a curve (not a surface), whose equations can be obtained by eliminating the two parameters from the equation of the system of surfaces and the three equations obtained by differentiating it with regard to the coordinates. Hence its equations cannot be determined by equating to zero a factor of the discriminant.

But if every surface of the system have a singular point, then *in general* its coordinates may be expressed as functions of the two parameters of the surface to which it belongs. Hence the locus of the singular points is a surface. It will be proved that it is a part of the locus of ultimate intersections. Hence its equations can be obtained by equating to zero a factor of the discriminant.

Let now $E = 0$ be the equation of the envelope-locus,
$C = 0$ the equation of the conic node locus,
$B = 0$ the equation of the biplanar node locus,
$U = 0$ the equation of the uniplanar node locus.

Now at any point on the locus of ultimate intersections—

(I.) *There may be one system of values of the parameters satisfying the fundamental equations.*

In this case there may be envelope, conic node, or biplanar node loci; and the results are given in the following table :—

Description of locus.	Factor of discriminant corresponding to locus.	Nature of intersection of surfaces represented by fundamental equations at a point on locus of ultimate intersections.
Envelope	E (Art. 9)	1 point (Art. 26)
Conic node locus	C^2 (Art. 10)	2 points (Art. 27)
Biplanar node locus ...	B^3 (Art. 11)	3 points (Art. 28)

(II.) *There may be more than one system of distinct values of the parameters satisfying the fundamental equations.*

In this case the effect of the distinct values is additive. Thus if there be p systems of values at a point on the envelope locus, the factor E would occur to the p^{th} power.

(III.) *Two or more systems of values of the parameters satisfying the fundamental equations may coincide.*

The results must be stated differently in the cases (α) where the degree in the parameters of the equation of the system of surfaces is greater than two; (β) where the degree in the parameters of the equation of the system of surfaces is two.

In the case (α) it will be shown that there may be envelope-loci, in which the envelope has stationary contact with each surface of the system; conic-node loci, which are also envelopes; biplanar node loci, in which the edge always touches the biplanar node locus; and uniplanar node loci. The results are given in the following table :—

Description of locus.	Factor of discriminant corresponding to locus.	Nature of intersection of surfaces represented by fundamental equations at a point on locus of ultimate intersections
Envelope locus having stationary contact with each surface of system	E^2 (Art. 13)	2 points (Art. 26)
Conic node locus, which is also an envelope	C^3 (Art. 14)	3 points (Art. 27)
Biplanar node locus with edge of biplanar node touching biplanar node locus	B^4 (Art. 15)	4 points (Art. 28)
Uniplanar node locus	U^6 (Art. 12)	6 points (Art. 29)

The case (β) always falls under the next case.

(IV.) *The values of the parameters satisfying the fundamental equations may become indeterminate.*

If the equation of the system of surfaces be of the second degree in the parameters, and the analytical condition hold which expresses that the fundamental equations are satisfied by two coinciding systems of values, then this condition requires to be specially interpreted. For now the second and third fundamental equations are of the first degree in the parameters, so that if they are satisfied by two coinciding systems of values, they must be indeterminate.

It is, however, possible to determine a *single* system of values of the parameters satisfying them. In this case the three surfaces represented by the fundamental equations intersect in a common curve (which is fixed for fixed values of the parameters) lying on the locus of ultimate intersections; whereas in the previous cases they intersect in a finite number of points lying on the locus of ultimate intersections.

The surface of the system, corresponding to the fixed values of the parameters, touches the locus of ultimate intersections along the above-mentioned curve.

In general there are *two* conic nodes of the system at every point of the locus of ultimate intersections. The parameters of the surfaces having the conic nodes are determined by two quadratic equations, called the parametric quadratics; and in general the roots of *each* parametric quadratic are *unequal*. If the roots of *both* parametric quadratics are *equal*, the two surfaces having conic nodes are replaced by one surface having a biplanar or uniplanar node.

If the parameters of *one* of the surfaces having a conic node become *infinite*, this surface may be considered to disappear, and there is but one conic node at each point of the locus of ultimate intersections.

If the parameters of *both* surfaces having conic nodes become *infinite*, both these surfaces may be considered to disappear, and the locus of ultimate intersections is an envelope locus (touching each surface of the system along a curve).

If the parameters of *both* surfaces having conic nodes become *indeterminate*, then there are at each point an infinite number of biplanar nodes, and each surface of the system has a binodal line lying on the locus of ultimate intersections.

The results are given in the following table:—

Description of locus.	Factor of discriminant corresponding to locus.	Both parametric quadratics have
Locus of two conic nodes	C^2 (Art. 19)	Their roots unequal
Biplanar node locus with edge of biplanar node touching biplanar node locus.	B^3 (Art. 21)	Their roots equal
Uniplanar node locus	U^4 (Art. 22)	Their roots equal
Locus of one conic node ..	C^2 (Art. 23)	One root infinite
Envelope locus..........	E^3 (Art. 24)	Both roots infinite
Binodal line locus	B^4 (Art. 25)	Their roots indeterminate

It will be noticed that when the equation of the system of surfaces is of the second degree in the parameters, and the condition holds which expresses that the fundamental equations are satisfied by two coinciding values of the parameters, there is a reduction in the number of factors of the discriminant corresponding to the singular point loci, the factors C^3, B^4, U^6 becoming C^2, B^3, U^4 respectively.

The explanation is as follows:—

The discriminant is formed by solving the second and third fundamental equations for the parameters, substituting each pair of values in the left-hand side of the first fundamental equation, multiplying the results together, and also multiplying by a rationalising factor. Now in the case where the degree of the equation in the parameters is the second, there is only *one* system of roots corresponding to the loci under consideration, whereas there are *two* when the degree in the parameters is above the second. Hence this accounts for a diminution in the number of factors when the degree in the parameters is the second.

But this diminution is partly counterbalanced by an increase due to the fact that the rationalising factor vanishes at every point on the locus of ultimate intersections, and consequently increases the number of factors corresponding to the singular point loci. The result of the two causes is what has been stated above.